

An application of harmonic analysis in number theory

An asymptotic for pairs of primes of the form $(p, p + \lfloor m^c \rfloor)$

Bora Çalim

(joint work with I. Iakovakis, S. Long, J. Moffatt, D. Wooton; as part
of Young Researchers in Mathematics Program at EPFL)

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Preliminary discussion

- Intuition: "additive structure and multiplicative structure of integers are roughly independent" (unless there is an obvious reason to the contrary)

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- Intuition: "additive structure and multiplicative structure of integers are roughly independent" (unless there is an obvious reason to the contrary)
- "Corollary": "Shifting primes should not change their behavior" (unless there is an obvious reason to the contrary: e.g. $p, p + 1$ cannot both be prime if $p > 2$)
- Examples: Twin prime conjecture (in strong form; $+2$ "is independent"), PNT in arithmetic progressions (restricting to an arithmetic progression "is independent"), Bateman-Horn conjecture (shifting by multiple polynomials is "independent"), etc.

- Consider the averages

$$A(N) = \frac{1}{NM} \sum_{n \leq N} \sum_{m \leq M} \Lambda(n) \Lambda(n + \lfloor m^c \rfloor),$$

where $\Lambda(n) = \log p$ if $n = p^k$ and 0 otherwise (von Mangoldt function), $c > 2$ is not an integer, $M = N^{1/c} \log^{-B} N$ for fixed (but arbitrary) B (this is for technical reasons).

Discussion of our result - 1

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- PNT says $\frac{1}{N} \sum_{n \leq N} \Lambda(n) \approx 1$, so in view of the independence heuristic we expect this average to be ≈ 1 also
- this average is related to pairs of primes $(p, p + \lfloor m^c \rfloor)$

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For any $B > 1$, there is some $C = C(B) > 0$ such that

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- Interpretation: "The independence heuristic holds in this case" (and we can quantify the error)
- Corollary: ∞ pairs of primes of the form $(p, p + \lfloor m^c \rfloor)$

How to attack?

$$A(N) = \frac{1}{NM} \sum_{n \leq N} \sum_{m \leq M} \Lambda(n) \Lambda(n + \lfloor m^c \rfloor),$$

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- Fourier analysis - good interaction with translations.

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- Interpretation: $\hat{f}(r)$ measures (roughly) how skewed the distribution of f is along arithmetic progressions of common difference close to N/r . Also: change of basis

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- This does not look more tractable...
- But: we have good bounds for $\hat{\Lambda}$ coming from analytic number theory, especially when r/N is not “close” to a “low-height” rational
- And: we can deal with $\frac{1}{M} \sum_{m \leq M} e\left(-\frac{r \lfloor m^c \rfloor}{N}\right)$ by several magical inequalities

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- “small” r : two cases - “very small” ($\lesssim \log^c N$) and the rest

“very small” r - no magic

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- On each interval, using PNT in the form $\left| \frac{1}{N} \sum_{n=0}^{N-1} \Lambda(n) - 1 \right| \leq \frac{C}{e^{d\sqrt{\log N}}}$

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- idea: add and subtract $1/rk$, triangle inequality, obtain $|\hat{\Lambda}(r)| \leq cN/\log^A N$ (there are more details)

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- Vinogradov's theorem: if $(a, q) = 1$, $|r/N - a/q| \leq 1/q^2$, then $|\hat{\Lambda}(r)| \leq C\left(\frac{N}{\sqrt{q}} + N^{\frac{4}{5}} + \sqrt{qN}\right)(\log N)^4$. (confirms the intuition)

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- Then use Vinogradov to obtain $|\hat{\Lambda}(r)| \leq cN/\log^{A/4} N$

- Recall: $\frac{1}{N^2} \sum_{|r|=1}^{N/\log^A N} \left| \hat{\Lambda}(r) \right|^2 \frac{1}{M} \sum_{m \leq M} e\left(-\frac{r \lfloor m^c \rfloor}{N}\right)$

Using Vinogradov

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- Hölder $\implies \leq \sup_{1 \leq r \leq N/\log^A N} \left| \frac{\hat{\Lambda}(r)}{N} \right|^{\frac{1}{k}}$

$$\cdot \left(\sum_{r=1}^{\frac{N}{\log^A N}} \left| \frac{\hat{\Lambda}(r)}{N} \right|^2 \right)^{1 - \frac{1}{2k}} \left(\sum_{r=1}^{\frac{N}{\log^A N}} \left| \frac{1}{M} \sum_{m \leq M} e\left(-\frac{r \lfloor m^c \rfloor}{N}\right) \right|^{2k} \right)^{\frac{1}{2k}}$$

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- First term $\leq c/\log^C N$ by Vinogradov, second term $\leq \log N$ by Parseval, the third term remains

The other term

- We want to control: $U = \sum_{r=1}^{\frac{N}{\log^A N}} \left| \frac{1}{M} \sum_{m \leq M} e\left(-\frac{r|m^c|}{N}\right) \right|^{2k}$

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- Poulias (2021) proved that the number of solutions to this is $\leq cM^{2k-c}$ when k is sufficiently large
- $U^{1/2k} \leq c \log^{BC/2k} N$, choose A suitably to get $\frac{1}{N^2} \sum_{|r|=1}^{N/\log^A N} \left| \hat{\Lambda}(r) \right|^2 \frac{1}{M} \sum_{m \leq M} e\left(-\frac{r \lfloor m^c \rfloor}{N}\right) \leq \log^{-C} N$

“large” r - First manipulations

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$$V = \frac{1}{N^2} \sum_{|r|=N/\log^A N}^{N/2} \left| \hat{\Lambda}(r) \right|^2 \frac{1}{M} \sum_{m \leq M} e\left(-\frac{r \lfloor m^c \rfloor}{N}\right)$$

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- So we reduce to bounding $\left| \frac{1}{M} \sum_{m \leq M} e\left(-\frac{r \lfloor m^c \rfloor}{N}\right) \right|$ uniformly in $|r|$ between $N/\log^A N$ and $N/2$

“large” r - Magic

- Recall: we want to bound $|\frac{1}{M} \sum_{m \leq M} e(-\frac{r \lfloor m^c \rfloor}{N})|$ uniformly in $N/\log^A N \leq |r| \leq N/2$

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- Combining, we get the main result.

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- Thank you!