# 1 Using Quadratic Fourier Analysis to Find 4-term Arithmetic Progressions

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**Abstract**

We give an inverse theorem for the Gowers $U^3$ norm on $\mathbb{F}_5^n$ and use it to prove the existence of many (proportional to the density of the set) 4 term arithmetic progressions with the same step size in subsets of $\mathbb{F}_5^n$.

## 1.1 Introduction

Let $1 \geq \alpha > 0$ be a real number. We aim to show the existence of 4 term arithmetic progressions in subsets $A \subset \mathbb{F}_5^n$ with density $\alpha$ for large enough $n$. Throughout the summary, $G$ will denote $\mathbb{F}_5^n$, and $N = |G|$.

**Theorem 1.** *Let $\alpha, \epsilon > 0$ be real numbers. Then there is an $n_0 = n_0(\alpha, \epsilon)$ with the following property. Suppose that $n > n_0(\alpha, \epsilon)$, and that $A \subseteq G$ is a set with density $\alpha$. Then there is some $d \neq 0$ such that $A$ contains at least $(\alpha^4 - \epsilon) N$ four-term arithmetic progressions with common difference $d$.*

Instead of working with the set $A \subset \mathbb{G}$, we will consider its characteristic function $1_A : \mathbb{G} \to \{0, 1\}$. The averages, the Fourier transform, and the Gowers uniformity norm of functions carry information about the number of arithmetic progressions in $A$. However, the techniques used to prove the existence of 3-APs cannot be directly generalized to 4-APs. We summarize these differences and introduce the required notions.

**Definition 2** $(\Lambda_3, \Lambda_4)$**.** *For $f_i : G \to [-1, 1]$ we define $\Lambda_3(f_1, f_2, f_3) = \mathbb{E}_{x,d} f_1(x) f_2(x + d) f_3(x + 2d)$, and $\Lambda_4(f_1, f_2, f_3, f_4)$ analogously.*

**Definition 3** (Gowers norms)**.** *The Gowers uniformity norm of $f : G \to \mathbb{R}$ for integer $d \geq 2$ is defined as follows*

$$\|f\|_{U^d}^{2^d} := \sum_{x, h_1, \ldots, h_d} \prod_{\omega_1, \ldots, \omega_d \in \{0,1\}} f(x + h_1 w_1 + \ldots + h_d \omega_d).$$

In $k = 3$ case

- The operator $\Lambda_3$ is controlled by the Gowers $U^2$-norm. Specifically for any three functions $f_1, f_2, f_3 : G \to [-1,1]$ we have
$$|\Lambda_3 (f_1, f_2, f_3)| \leqslant \inf_{i=1,2,3} \|f_i\|_{U^2}.$$
- (Gowers inverse theorem) If the Gowers $U^2$-norm of a function $f : G \to [-1,1]$ is large, $f$ must have a large Fourier coefficient:
$$\|f\|_{U^2} \geqslant \delta \quad \Rightarrow \quad \|\widehat{f}\|_\infty \geqslant \delta^2.$$

The first item is directly generalized, while the second item is not. The following proposition and example illustrate this.

**Proposition 4.** *Let $f_1, \ldots, f_4 : G \to [-1,1]$ be any four functions. Then we have*
$$|\Lambda_4 (f_1, \ldots, f_4)| \leqslant \inf_{i=1,\ldots,4} \|f_i\|_{U^3}.$$

**Example 5.** *There is a function $f : G \to \mathbb{C}$ with $\|f\|_\infty \leqslant 1$ such that $\|f\|_{U^3} = 1$, but such that $\|\widehat{f}\|_\infty \leqslant N^{-1/2}$. Namely $f = w^{x^T x}$.*

Instead, we find that f has significant correlation with a quadratic phase:

**Theorem 6.** *Suppose that $f : G \to [-1,1]$ is a function for which $\|f\|_{U^3} \geqslant \delta$. Then there is a matrix $M \in \mathfrak{M}_n(\mathbb{F}_5)$ and a vector $r \in \mathbb{F}_5^n$ so that*
$$\left|\mathbb{E}_{x \in G} f(x) \omega^{x^T M x + r^T x}\right| \gg_\delta 1.$$

# 2 Proof of Theorem 6

There are 3 steps in proving a function $f$ with large $U^3$ norm correlates with a quadratic phase $w^{x^T M x + r^T x}$. Throughout, $|G| \gg_\delta 1$ whenever needed, $f : G \to [-1,1]$, $\|f\|_{U^3} \geqslant \delta$, $M$ denotes an $n \times n$ matrix with entries from $\mathbb{F}_5$, $b$ denotes a vector in $\mathbb{F}_5^n$, and $\Delta(f;h)(x) = f(x)f(x-h)$ is a "multiplicative derivative".

The first step is to show that the derivative of $f$ obeys a "weak linearity" property: There is a function $\phi : G \to \widehat{G}$ and $S \subseteq G$ with $|S| \gg_\delta |G|$ such that

1. $|\Delta(f;h)^\wedge(\phi(h))| \gg_\delta 1$ for all $h \in S$

2. There are $\gg_\delta |G|^3$ quadruples $(s_1, s_2, s_3, s_4) \in S^4$ such that $s_1 + s_2 = s_3 + s_4$ and $\phi(s_1) + \phi(s_2) = \phi(s_3) + \phi(s_4)$.

The second step is to show that this weak linearity property implies a stronger linearity property: If $\phi : G \to \widehat{G}$, $S \subseteq G$ satisfy the conclusions 1 and 2 of the previous step, then there is some linear function $\psi(x) = Mx + b$ such that $\psi(x) = \phi(x)$ for $\gg_\delta |G|$ values of $x \in S$. We give a sketch of the proof of this step.

Consider $\Gamma = \{(h, \phi(h)) : h \in S\}$. By conclusion 2 of the first step, we can use the Balog-Szemerédi-Gowers theorem to find some $\Gamma' \subseteq \Gamma$ such that $|\Gamma'| \gg_\delta |\Gamma| \gg_\delta |G|$ and $|\Gamma' + \Gamma'| \ll_\delta |\Gamma'|$. Identifying $G \times \widehat{G}$ with $\mathbb{F}_5^{2n}$, by Freiman's theorem, we can find a subspace $H \subseteq \mathbb{F}_5^{2n}$ containing $\Gamma'$ such that $|H| \ll_\delta |\Gamma'| \ll_\delta |G|$.

Consider the canonical projection $\pi : H \to G$ to the first factor, and let $S' = \pi(\Gamma')$, so that $|\pi(H)| \geq |S'| \gg_\delta |G|$. By the rank-nullity theorem, it follows that $\dim \ker(\pi) \ll_\delta 1$. Let $H' = (\ker(\pi))^\perp$, so that $H = \bigcup_{x \in \ker(\pi)} (H' + x)$, where the union is disjoint and taken over $\ll_\delta 1$ elements. Observe that $\pi$ is injective on each of the cosets in the union. By the pigeonhole principle, there is some $x$ such that $|(x + H') \cap \Gamma'| \gg_\delta |\Gamma'| \gg_\delta |G|$. Let $\Gamma'' = (x + H') \cap \Gamma'$ and $S'' = \pi(\Gamma'')$, $V = \pi(x + H')$. Then $\psi : V \to \widehat{G}$ given by the composition of $\pi^{-1}$ and the canonical projection to the second factor is an affine map, so $\psi(x) = Mx + b$ for some $M, b$. It can be seen that $\psi(x) = \phi(x)$ for all $x \in S''$, so the proof is complete.

Combining the two steps, we can find some $M$, $b$ such that

$$\mathbb{E}_h \left|\Delta(f;h)^\wedge(Mh + b)\right|^2 \gg_\delta 1.$$

It turns out that a Matrix $M$ satisfying the above bound is approximately symmetric in a precise sense: If

$$\mathbb{E}_h \left|\Delta(f;h)^\wedge(Mh + b)\right|^2 \gg_\delta 1,$$

Then $\operatorname{rank}(M) \ll_\delta 1$.

From this we can recover a fully symmetric matrix $M'$, which gives theorem 6.

## 3 Arithmetic Regularity for $U^3$

In this section, the main objective is to decompose a function $f : \mathbb{G} \to [-1, 1]$ into three parts. The first one, $\mathbb{E}(f \mid \mathcal{B})$, is constant on certain sets, the second one is the error term in the sense of having a small $L_2$ norm, and the third has a small $U^3$ norm.

**Definition 7** (Factors, Conditional Expectation, Rank of a Quadratic Factor)**.** *Let $\phi_1, \ldots, \phi_k : G \to G$ be any functions. The $\sigma$-algebra, $\mathcal{B}$, generated by the sets (atoms) of the form $\{x \in G \mid \phi_1(x) = c_1, \ldots, \phi_k(x) = c_k\}$ are called a factor. The conditional expectation of $f$ is defined as*

$$\mathbb{E}(f \mid \mathcal{B})(x) := \mathbb{E}_{x \in \mathcal{B}(x)} f(x)$$

*where $\mathcal{B}(x)$ is the atom of $\mathcal{B}$ containing $x$. If all the functions $\phi_i(x)$ $i \leq k$ are of the form $r_i^T x$ for some $r_i \in G$ the factor $\mathcal{B}$ generated by $\phi_i, i \leq k$ is called a linear factor of complexity at most $k$.*

*Let $i \leq d_i, r_i \in G$ and $M_j, j \leq d_2$ be symmetric matrices in $\mathcal{M}_n(G)$. Let $\mathcal{B}_1$ be the factor generated by the linear functions $\phi_i(x) = r_i^T x$; and $\mathcal{B}_2$ be the factor generated by $\phi_i(x) = r_i^T x, i \leq d_1$ and $\psi_j(x) = x^T M_j x$, $j \leq d_1$. $\mathcal{B}_2$ is a refinement of $\mathcal{B}_1$. $(\mathcal{B}_1, \mathcal{B}_2)$ is called a factor of complexity $(d_1, d_2)$. We say that $(\mathcal{B}_1, \mathcal{B}_2)$ has rank at least $r$ if for all nontrivial linear combinations of $M_1, \ldots, M_{d_2}$ has rank at least $r$.*

With the following lemma, we write any function $f : G \to [-1, 1]$ as a sum of a measurable function with respect to a quadratic factor and two error terms that are small, respectively, in $L^2$ and $U^3$. The strength of the lemma is to make $\|f_3\|_{U^3}$ arbitrarily small by choosing a suitable growth function $\omega_2$ with the cost of making the complexity higher.

**Lemma 8.** *Let $\delta > 0$ be a parameter, and let $\omega_1, \omega_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be arbitrary growth functions (which may depend on $\delta$ ). Let $n > n_0(\delta, \omega_1, \omega_2)$ be sufficiently large, and let $f : G \to [1, 1]$ be a function. Let $\left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$ be a quadratic factor of complexity $\left(d_1^{(0)}, d_2^{(0)}\right)$. Then there is a quadratic factor $(\mathcal{B}_1, \mathcal{B}_2)$ with the following properties: $(\mathcal{B}_1, \mathcal{B}_2)$ refines $\left(\dot{\mathcal{B}}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$; the complexity of $(\mathcal{B}_1, \mathcal{B}_2)$ is at most $(d_1, d_2)$, where*

$$d_1, d_2 \leqslant C\left(\delta, \omega_1, \omega_2, d_1^{(0)}, d_2^{(0)}\right),$$

*for some fixed function $C$; the rank of $(\mathcal{B}_1, \mathcal{B}_2)$ is at least $\omega_1(d_1+d_2)$; there is a decomposition $f = f_1 + f_2 + f_3$, where*

$$f_1 := \mathbb{E}(f \mid \mathcal{B}_2),$$
$$\|f_2\|_2 \leqslant \delta,$$
$$\|f_3\|_{U^3} \leqslant 1/\omega_2(d_1+d_2).$$

# 4 Main Theorem

*To understand $\mathcal{B}_2$ measurable functions, i.e., functions that are constant on the atoms of $\mathcal{B}_2$ with complexity $(d_1, d_2)$, we study functions on the configuration space $\mathbb{F}_5^{d_1} \times \mathbb{F}_5^{d_2}$. We take $r_1, \ldots, r_{d_1}$ linearly independent and define $\Gamma(x) := (r_1^T, \ldots, r_{d_1}^T)$ and $\Phi(x) := (x^T M_1 x, \ldots, x_{d_2}^T M_{d_2} x)$.*

*Proof of theorem 1.* We apply theorem 8 to $1_A$ to obtain a decomposition $1_A = f_1 + f_2 + f_3$ such that the quadratic factor $(\mathcal{B}_1, \mathcal{B}_2)$ is with complexity $(d_1, d_2)$ $d_i \leq d_0(\alpha, \epsilon)$ and the rank $r$ is such that

$$r \geq 100(\log(1/\epsilon) + \log(1/\alpha) + d_1 + d_2).$$

The parameter $\delta$ and $\omega$ (which only depends on $\alpha$ and $\epsilon$ justifying the bound for $d_0$) will be specified afterwards. We define the $n - d_1$ dimensional space $H := \langle r_1, \ldots, r_{d_1} \rangle^\perp$, and $\mu_H$ to be the normalised measure $\mu_H : 1_H / \mathbb{E} 1_H$. To prove the theorem, we show

$$\mathbb{E}_{x,d} 1_A(x) 1_A(x+d) 1_A(x+2d) 1_A(x+3d) \mu_H(d) > (\alpha^4 - \epsilon).$$

The left-hand side of the above expression splits into 81 parts after the substitution $1_A = f_1 + f_2 + f_3$.
*Claim 1.* The 65 terms containing $f_2$ has contribution $\leq \epsilon/200$.
*Claim 2.* The 65 terms containing $f_3$ has contribution $\leq \epsilon/200$.

*Proof.* Suppose that $g_1 = f_3$, the other cases are similar. We write the term as

$$\mathbb{E}_{x,d} g_1(x) g_2(x+d) g_3(x+2d) g_4(x+3d) \mu_H(d) \tag{1}$$

where $g_2, g_3, g_4$ are one of the $f_1, f_2, f_3$. We make the observation

$$1_H(d) = \sum_t 1_{t+H}(x) 1_{t+H}(x+2d)$$

where the sum is over all cosets of $H$ in $G$. By proposition 4

$$\mathbb{E}_{x,d} g_1(x) g_2(x+d) 1_{t+H}(x+d) g_3(x+2d) 1_{t+H}(x+2d) g_4(x+3d) \leq \|f_3\|_{U^3} \leq 1/\omega_2(d_1+d_2).$$

Hence we bound (1) by $< 5^{2d_1}/\omega(d_1+d_2)$. Provided that $\omega(m) \geq 5^{m+4}/\epsilon$. □

*Claim 3.* As $f$ is a $\mathcal{B}_2$ measurable function we define $\mathbf{f}_1 : \mathbb{F}_5^{d_1} \times \mathbb{F}_5^{d_2}$ such that $f_1(x) = \mathbf{f}_1(\Gamma(x), \phi(x))$ for all $x \in G$. Since the size of the factors are not equal, we have

$$\begin{aligned} &\mathbb{E}_{x,d} f_1(x) f_1(x+d) f_1(x+2d) f_1(x+3d) \mu_H(d) \\ &= \mathbb{E}_{\substack{a\in\mathbb{F}_5^{d_1}, b^{(1)},\dots,b^{(4)}\in\mathbb{F}_5^{d_2} \\ b^{(1)}-3b^{(2)}+3b^{(3)}-b^{(4)}=0}} \mathbf{f}_1(a,b^{(1)}) \mathbf{f}_1(a,b^{(2)}) \mathbf{f}_1(a,b^{(3)}) \mathbf{f}_1(a,b^{(4)}) \\ &+ O(5^{2d_1+3d_2-r/2}). \end{aligned}$$

The constraints on $a$ and $b$ is a result of two facts: $d \in H$ and $\Phi(x) - 3\Phi(x+d) + 3\Phi(x+2d) - \Phi(x+3d) = 0$.

$$\begin{aligned} (5^{-2d_1-3d_2} + O(5^{-r/2})) \sum_{a\in\mathbb{F}_5^n} \sum_{\substack{a\in\mathbb{F}_5^{d_1}, b^{(1)},\dots,b^{(4)}\in\mathbb{F}_5^{d_2} \\ b^{(1)}-3b^{(2)}+3b^{(3)}-b^{(4)}=0}} & \mathbf{f}_1(a,b^{(1)}) \mathbf{f}_1(a,b^{(2)}) \\ & \times \mathbf{f}_1(a,b^{(3)}) \mathbf{f}_1(a,b^{(4)}) \\ \geq (5^{-2d_1-3d_2} + O(5^{-r/2}))(\mathbb{E}_{(a,b)\in\mathbb{F}_5^{d_1}\times\mathbb{F}_5^{d_2}} & \mathbf{f}_1(a,b))^4. \end{aligned}$$

The last line follows from two applications of Cauchy-Schwarz.
*Claim 4.* $\mathbb{E}_{(a,b)\in\mathbb{F}_5^{d_1}\times\mathbb{F}_5^{d_2}} \mathbf{f}_1(a,b) = \alpha(1 + O(5^{d_1+d_2-r/2}))$. This claim is a result of the fact that atoms are close in size. After some calculations, the theorem follows from these four claims.

□

# References


[G] Green, B. *Montréal Notes on Quadratic Fourier Analysis.* Proceedings of the CRM-Clay Conference on Additive Combinatorics, Montréal 2006.



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