# Using Quadratic Fourier Analysis to Find 4-term Arithmetic Progressions

(After Green)

Bora Çalım and Nihan Tanisali

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## Theorem (Main Theorem)

*Let* $\alpha, \epsilon > 0 \in \mathbb{R}$. *Then* $\exists n_0 = n_0(\alpha, \epsilon)$ *with the following property: Suppose that* $n > n_0$ *and* $A \subseteq \mathbb{F}_5^n$ *has density* $\alpha$. *Then* $\exists d \neq 0$ *s.t.* $A$ *contains* $\geq (\alpha^4 - \epsilon)N$ *4APs with common difference d* ($N = |\mathbb{F}_5^n| = 5^n$).

- Best possible result (consider random set of density $\alpha$)
- $\mathbb{F}_5^n$ is a useful model setting (subspaces etc.)

# Preliminaries

- Throughout, $G = \mathbb{F}_5^n$, $N = |G| = 5^n$
- $\hat{f}(r) = \mathbb{E}_{x \in G} f(x)\omega^{r^T x}, \omega = e^{\frac{2\pi i}{5}}$
- Haar measure in $G$, counting measure in $\widehat{G}$
- Parseval: $\|f\|_2 = \|\hat{f}\|_2$

# First definitions

- To analyze 3APs we can use ordinary Fourier Analysis (e.g. Roth). Hard to generalize
- Approach based on Gowers norms, easier to generalize
- Multilinear forms $\Lambda_3$ and $\Lambda_4$, defined on $f_i : G \to [-1, 1]$
- $\Lambda_3(f_1, f_2, f_3) = \mathbb{E}_{x,d} f_1(x) f_2(x + d) f_3(x + 2d)$
- $\Lambda_4(f_1, f_2, f_3, f_4) = \mathbb{E}_{x,d} f_1(x) f_2(x + d) f_3(x + 2d) f_4(x + 3d)$

# Gowers norms

- Gowers norms $U^2$ and $U^3$, for $f : G \to [-1,1]$:

$$\begin{aligned}\|f\|_{U^2} &= (\mathbb{E}_{y_1,y_2,y_1',y_2'} f(y_1+y_2)f(y_1+y_2')f(y_1'+y_2)f(y_1'+y_2'))^{1/4} \\ &= (\mathbb{E}_{x,h_1,h_2} f(x)f(x+h_1)f(x+h_2)f(x+h_1+h_2))^{1/4}\end{aligned}$$

$$\begin{aligned}||f||_{U^3}^8 &= \mathbb{E}_{y_1,y_2,y_3,y_1',y_2',y_3'} f(y_1+y_2+y_3)f(y_1+y_2+y_3')f(y_1+y_2'+y_3) \\ &\times f(y_1'+y_2+y_3)f(y_1+y_2'+y_3')f(y_1'+y_2+y_3') \\ &\times f(y_1'+y_2'+y_3)f(y_1'+y_2'+y_3') \\ &= \mathbb{E}_{x,h_1,h_2,h_3} f(x)f(x+h_1)f(x+h_2)f(x+h_3)f(x+h_1+h_2) \\ &\times f(x+h_1+h_3)f(x+h_2+h_3)f(x+h_1+h_2+h_3)\end{aligned}$$

- "average over parallelepipeds"

# A Brief Look at the 3AP case

- $\Lambda_3$ is controlled by $U^2$:
- $|\Lambda_3(f_1, f_2, f_3)| \leq \inf_{i=1,2,3} \|f_i\|_{U^2}$
- Large $U^2$ norm implies a large FC:
- $\|f\|_{U^2} \geq \delta \implies \|\hat{f}\|_\infty \geq \delta^2$

# Attempt to generalize to 4APs

- First can be generalized: $\Lambda_4$ is controlled by $U^3$:
- $|\Lambda_4(f_1, f_2, f_3, f_4)| \leq \inf_{i=1,2,3,4} \|f_i\|_{U^3}$
- Second cannot: $f(x) = \omega^{x^T x}$.
- $\|f\|_{U^3} = 1$, $\|\hat{f}\|_\infty \leq N^{-1/2}$
- Instead of a large FC, find correlation between $f$ and a quadratic phase

We want to prove:

### Theorem (inverse result for $U^3$ norm on $\mathbb{F}_5^n$)

*Suppose that* $f : \mathbb{F}_5^n \to [-1, 1]$ *is a function for which* $\|f\|_{U^3} \geqslant \delta$. *Then there is a matrix* $M \in \mathfrak{M}_n(\mathbb{F}_5)$ *and a vector* $r \in \mathbb{F}_5^n$ *so that*

$$\left|\mathbb{E}_{x \in G} f(x) \omega^{x^T M x + r^T x}\right| \gg_\delta 1.$$

"If the $U^3$ norm of $f$ is large, then $f$ correlates with a quadratic function"

# Steps in proving the inverse result for $U^3$

- If $\|f\|_{U^3} \geq \delta$, then $\Delta(f;h)(x) = f(x)f(x-h)$ has some "weak linearity"
- This weak linearity implies stronger linearity
- $\Delta(f;h)$ correlates with a linear phase $\mathbb{E}_h|\Delta(f;h)^\wedge(Mh+b)|^2 \gg_\delta 1$
- $M$ is necessarily "almost symmetric" and WLOG we can take $M$ symmetric
- "Integration" of the previous statement to obtain the result

# Step 1 - weak linearity from large $U^3$ norm

Precise statement: If $\|f\|_{U^3} \geq \delta$ and $|G| \gg_\delta 1$, then $\exists \phi : G \to \widehat{G}, S \subseteq G$ with $|S| \gg_\delta |G|$ such that the following hold:

1. $|\Delta(f;h)^\wedge(\phi(h))| \gg_\delta 1$ for all $h \in S$
2. There are $\gg_\delta |G|^3$ quadruples $(s_1, s_2, s_3, s_4) \in S^4$ s.t. $s_1 + s_2 = s_3 + s_4$ and $\phi(s_1) + \phi(s_2) = \phi(s_3) + \phi(s_4)$

# Sketch of step 1

- From $\|f\|_{U^3} \geq \delta$, algebra, Hölder, Parseval
$\implies \mathbb{E}_h \|\Delta(f;h)^\wedge\|_8^8 \geq \delta^{24}$

- Samorodnitsky $\implies LHS =$

$$\sum_{r_1+r_2=r_3+r_4} \mathbb{E}_{h_1+h_2=h_3+h_4} |\Delta(f;h_1)^\wedge(r_1)|^2 \cdots |\Delta(f;h_4)^\wedge(r_4)|^2 \tag{1}$$

- For $h \in G$, let $\Phi(h) = \{r : |\Delta(f;h)^\wedge(r)| \geq \delta^{50}\}$

- Contributions to (1) where $r_i \notin \Phi(h_i)$ for some $i$ is small

# Sketch of step 1

- We have:

$$\sum_{r_1+r_2=r_3+r_4} \mathbb{E}_{h_1+h_2=h_3+h_4} |\Delta(f;h_1)^\wedge(r_1)|^2 \cdots |\Delta(f;h_4)^\wedge(r_4)|^2 \gg_\delta 1$$

- Restrict to $r_i \in \Phi(h_i)$ for all $i$ with small loss
- $\exists \gg_\delta N^3$ octuples $(h_1, r_1, \ldots, h_4, r_4)$ s.t. $h_1 + h_2 = h_3 + h_4$, $r_1 + r_2 = r_3 + r_4$, $r_i \in \Phi(h_i)$ for all $i$
- Octuples where $h_i$ not all distinct are $\ll_\delta N^2$, so we can restrict to $h_i$ all distinct with small loss

# Completing step 1

- We have: $\exists \gg_\delta N^3$ octuples $(h_1, r_1, \ldots, h_4, r_4)$ s.t. $h_1 + h_2 = h_3 + h_4$, $r_1 + r_2 = r_3 + r_4$, $r_i \in \Phi(h_i)$ for all $i$, $h_i$'s all distinct
- Take $S = \{h : \Phi(h) \neq \emptyset\}$ ($|S| \gg_\delta |G|$ to allow $\gg_\delta N^3$ octuples)
- Choose $\phi(h)$ from $\Phi(h)$ randomly ($|\Phi(h)| \ll_\delta 1$ by Parseval)
- $\exists \phi$ s.t. $\exists \gg_\delta |G|^3$ quadruples $(h_1, \ldots, h_4)$ s.t. $h_1 + h_2 = h_3 + h_4$, $\phi(h_1) + \phi(h_2) = \phi(h_3) + \phi(h_4)$ and $|\Delta(f; h)^\wedge(\phi(h))| \geq \delta^{50}$ for all h

# Step 2 - stronger linearity from weak linearity

- Precise statement: If $\phi : G \to \widehat{G}$ satisfies the conclusions (1) and (2) of step 1 (i.e., $\exists S \subseteq G, |S| \gg_\delta |G|$ s.t. $|\Delta(f;h)^\wedge(\phi(h))| \gg_\delta 1$ for all $h \in S$; and there are $\gg_\delta |G|^3$ quadruples $(s_1, s_2, s_3, s_4) \in S^4$ s.t. $s_1 + s_2 = s_3 + s_4$ and $\phi(s_1) + \phi(s_2) = \phi(s_3) + \phi(s_4)$), then $\exists \psi(x) = Mx + b$ s.t. $\psi(x) = \phi(x)$ for $\gg_\delta |G|$ values of $x \in S$.

# Sketch of step 2

- Consider $\Gamma = \{(h, \phi(h)) : h \in S\}$. Conclusion 2 & Balog-Szemerédi-Gowers $\implies \exists \Gamma' \subseteq \Gamma$ s.t. $|\Gamma'| \gg_\delta |\Gamma|$ and $|\Gamma' + \Gamma'| \ll_\delta |\Gamma'|$ (in particular $|\Gamma'| \gg_\delta |G|$)

- Freiman's thm. in $\mathbb{F}_p^n \implies \exists$ subspace $H \leq G \times \widehat{G} \cong \mathbb{F}_5^{2n}$ s.t. $\Gamma' \subseteq H$ and $|H| \ll_\delta |\Gamma'|$ (in particular $|H| \ll_\delta |G|$)

- Consider $\pi : H \to G$, $(a, b) \to a$ (projection), $S' = \pi(\Gamma')$

- $|\pi(H)| \geq |S'| \gg_\delta |G|$, rank-nullity $\implies \dim\ker(\pi) \ll_\delta 1$

# Sketch of step 2

- Consider $H' = (\ker(\pi))^{\perp}$. $H = \bigcup_{x \in \ker(\pi)} (x + H')$ ($\ll_\delta 1$ cosets in the disjoint union)
- $\pi$ injective on each coset $x + H'$
- Pigeonhole $\implies \exists x$ s.t. $|(x + H') \cap \Gamma'| \gg_\delta |\Gamma'| \gg_\delta |G|$

# Completing step 2

- $\Gamma'' = (x + H') \cap \Gamma'$, $S'' = \pi(\Gamma'')$, $V = \pi(x + H')$
- Consider $\psi : V \to \widehat{G}$, $v \to \pi^{-1}(v) = (a, b) \to b$
- $\psi$ is affine (so $\psi(x) = Mx + b$) and agrees with $\phi$ on $S''$, and $|S''| \gg_\delta |G|$

# Summary of what we have so far

- $|\Delta(f;h)^\wedge(\phi(h))| \gg_\delta 1$ for all $h \in S$, $|S| \gg_\delta |G|$
- $\phi(h) = \psi(h) = Mh + b$ for all $h \in S'' \subseteq S$, $|S''| \gg_\delta |G|$
- Corollary: $\mathbb{E}_h |\Delta(f;h)^\wedge(Mh+b)|^2 \gg_\delta 1$
- Next step: "symmetry argument" to obtain $\mathrm{rank}(M - M^T) \ll_\delta 1$

# Step 3 - symmetry argument

- Precise statement: If $\mathbb{E}_h|\Delta(f;h)^\wedge(Mh+b)|^2 \gg_\delta 1$, then $\operatorname{rank}(D) \ll_\delta 1$, where $D = M - M^T$

- First step of proof: Expand hypothesis and change variables to obtain $\exists g_z : G \to \mathbb{C}$, $\|g_z\|_\infty \leq 1$ s.t.

$$|\mathbb{E}_z \mathbb{E}_{x,y} g_z(x)\overline{g_z(y)}\omega^{x^T Dy}| \gg_\delta 1$$

# Sketch of step 3

- We have: $\exists g_z : G \to \mathbb{C}$, $\|g_z\|_\infty \leq 1$ s.t.

$$|\mathbb{E}_z \mathbb{E}_{x,y} g_z(x)\overline{g_z(y)}\omega^{x^T Dy}| \gg_\delta 1$$

- Pigeonhole, algebra, $D^T = -D \implies \exists g : G \to \mathbb{C}$, $\|g\|_\infty \leq 1$,

$$|\mathbb{E}_{x,y}\overline{g(x)}g(y)\omega^{(Dx)^T y}| \gg_\delta 1$$

- Observe: LHS $= |\mathbb{E}_x \overline{g(x)}\widehat{g}(Dx)|$

# Sketch of step 3

- We have: $\exists g : G \to \mathbb{C}$, $\|g\|_\infty \leq 1$ s.t. $|\mathbb{E}_x \overline{g(x)} \widehat{g}(Dx)| \gg_\delta 1$
- Since $\|g\|_\infty \leq 1$, $|\mathbb{E}_x \widehat{g}(Dx)| \gg_\delta 1$ so $\exists \gg_\delta |G|$ $x$'s s.t. $|\widehat{g}(Dx)| \gg_\delta 1$
- But Parseval gives $\exists \ll_\delta 1$ $r$'s s.t. $|\widehat{g}(r)| \gg_\delta 1$
- Thus $D$ takes $\gg_\delta |G|$ elements to $\ll_\delta 1$ elements, so $\mathrm{rank}(D) \ll_\delta 1$

# Getting full symmetry

- We have: $\|f\|_{U^3} \geq \delta \implies \mathbb{E}_h |\Delta(f;h)^\wedge(Mh+b)|^2 \gg_\delta 1$ with $\mathrm{rank}(M - M^T) \ll_\delta 1$
- Derivative of quadratic phase is a *symmetric* linear phase, so we want $M$ symmetric
- Fortunately, we can assume $M$ is symmetric (by a probabilistic argument over cosets of $\ker(M - M^T)$, which there are $\ll_\delta 1$, we can take $M := \frac{1}{2}(M + M^T)$)

# Last step ("integration")

- We have: $\mathbb{E}_h|\Delta(f;h)^\wedge(Mh+b)|^2 \gg_\delta 1$ with $M$ symmetric
- We want to "integrate" this to get the inverse result $\left|\mathbb{E}_{x\in G} f(x)\omega^{x^T Mx + r^T x}\right| \gg_\delta 1$
- Expanding the hypothesis, changes of variable, algebra $\implies \mathbb{E}_{h,x,k} g_1(x) g_2(x-h) g_3(x-k) g_4(x-h-k) \gg_\delta 1$ with $g_1(x) = f(x)\omega^{\frac{1}{2}x^T Mx}$, $\|g_i\|_\infty \leq 1$
- LHS $= \sum_r \widehat{g_1}(r)\widehat{g_2}(-r)\widehat{g_3}(-r)\widehat{g_4}(r)$

# Completing the proof of the inverse result

- We have: $\sum_r \widehat{g_1}(r)\widehat{g_2}(-r)\widehat{g_3}(-r)\widehat{g_4}(r) \gg_\delta 1$
- Hölder $\implies \|\widehat{g_1}\|_4 \gg_\delta 1$
- Then, by Cauchy-Schwarz and Parseval,

$$1 \ll_\delta \|\widehat{g_1}^4\|_1 \leq \|\widehat{g_1}^2\|_1 \|\widehat{g_1}^2\|_\infty \leq \|\widehat{g_1}^2\|_2 \|\widehat{g_1}^2\|_\infty \leq \|\widehat{g_1}^2\|_\infty$$

- $\|\widehat{g_1}\|_\infty \gg_\delta 1$, which means

$$\exists r \text{ s.t. } |\mathbb{E}_x f(x)\omega^{\frac{1}{2}x^T M x + r^T x}| \gg_\delta 1$$

# Objectives and Outline

- Definitions: factors, quadratic factors etc.
- Energy increment
- Versions of arithmetic regularity
- The main theorem.

# Factors

## Definition (Factors, Linear Factors, Conditional Expectation)

- Let $\phi_1, \ldots, \phi_k : G \to G$ be any functions. The $\sigma$-algebra, $\mathcal{B}$, generated by the sets (atoms) of the form $\{x \in G \mid \phi_1(x) = c_1, \ldots, \phi_k(x) = c_k\}$ are called a factor.
- If all the functions $\phi_i(x)$ $i \leq k$ are of the form $r_i^T x$ for some $r_i \in G$ the factor $\mathcal{B}$ generated by $\phi_i, i \leq k$ is called a linear factor of complexity at most $k$.
- The conditional expectation of $f$ is defined as

$$\mathbb{E}(f \mid \mathcal{B})(x) := \mathbb{E}_{x \in \mathcal{B}(x)} f(x)$$

where $\mathcal{B}(x)$ is the atom of $\mathcal{B}$ containing $x$.

## Definition (Rank of a Quadratic Factor)

- Let $i \leq d_i, r_i \in G$ and $M_j, j \leq d_2$ be symmetric matrices in $\mathcal{M}_n(G)$.
- Let $\mathcal{B}_1$ be the factor generated by the linear functions $\phi_i(x) = r_i^T x$; and $\mathcal{B}_2$ be the factor generated by $\phi_i(x) = r_i^T x, i \leq d_1$ and $\psi_j(x) = x^T M_j x$, $j \leq d_1$.
- $\mathcal{B}_2$ is a refinement of $\mathcal{B}_1$. $(\mathcal{B}_1, \mathcal{B}_2)$ is called a factor of complexity $(d_1, d_2)$.
- We say that $(\mathcal{B}_1, \mathcal{B}_2)$ has rank at least $r$ if for all nontrivial linear combinations of $M_1, \ldots, M_{d_2}$ has rank at least $r$.

# Energy Increment

## Lemma

*Let* $(\mathcal{B}_1, \mathcal{B}_2)$ *be a quadratic factor of complexity at most* $(d_1, d_2)$*, and let* $f : \mathbb{F}_5^n \to [-1, 1]$ *be a function such that*

$$\|f - \mathbb{E}(f \mid \mathcal{B}_2)\|_{U^3} \geq \delta.$$

*Then exists a refinement* $(\mathcal{B}_1', \mathcal{B}_2')$ *of* $(\mathcal{B}_1, \mathcal{B}_2)$ *of complexity at most* $(d_1 + 1, d_2 + 1)$ *such that we have the energy increment*

$$\|\mathbb{E}(f \mid \mathcal{B}_2')\|_2^2 \geq \|\mathbb{E}(f \mid \mathcal{B}_2)\|_2^2 + c(\delta)$$

*where* $c : (0, 1) \to \mathbb{R}^+$ *is some non-decreasing function of* $\delta$*.*

# Pythagoras Theorem

## Theorem

*Suppose that $\mathcal{B}, \mathcal{B}'$ are two $\sigma$-algebras on $\mathbb{F}_5^n$ such that $\mathcal{B}'$ refines $\mathcal{B}$. Let $f : \mathbb{F}_5^n \to [-1, 1]$ be any function. Then*

$$\left\|\mathbb{E}\left(f \mid \mathcal{B}'\right)\right\|_2^2 = \|\mathbb{E}(f \mid \mathcal{B})\|_2^2 + \left\|\mathbb{E}\left(f \mid \mathcal{B}'\right) - \mathbb{E}(f \mid \mathcal{B})\right\|_2^2.$$

## Proof Idea.

The proof is based on the following equality

$$(a + ld)^2 k + (a - kd)^2 l = a^2(k + l) + k(ld)^2 + l(kd)^2.$$

We assumed there is 1 atom $B$ of $\mathcal{B}$ and $\mathbb{E}(f \mid \mathcal{B}) = a$, and 2 atoms $B_1$ and $B_2$ of sizes $k$ and $l$ respectively. Then

$$\mathbb{E}(f \mid \mathcal{B}') = \begin{cases} a - ld & \text{if } x \in B_1 \\ a - kd & \text{if } x \in B_2. \end{cases}$$

□

## Proof of the energy increment.

- $g(x) := f(x) - \mathbb{E}(f - f \mid \mathcal{B}_2)$.
- By the inverse result for the $U^3$ norm, there exists non decreasing $c$ such that
$$|\mathbb{E}_x g(x)\omega^{x^T Mx + r^T x}| \geq c(\delta).$$
- The linear part and the quadratic part $r^T x$, $x^T Mx$ induces a quadratic factor $(\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2)$ of complexity $(1, 1)$.
- $x^T Mx + r^T x$ is $\tilde{\mathcal{B}}_2$-measurable. Hence
$$\mathbb{E}_x g(x)\omega^{x^T Mx + r^T x} = \mathbb{E}_x \mathbb{E}(g \mid \tilde{\mathcal{B}}_2)(x)\omega^{x^T Mx + r^T x}.$$
- $$\|\mathbb{E}(g \mid \tilde{\mathcal{B}}_2)\|_1 \geq c(\delta).$$
- Define $\mathcal{B}_1' := \mathcal{B}_1 \vee \tilde{\mathcal{B}}_1$ and $\mathcal{B}_2' := \mathcal{B}_2 \vee \tilde{\mathcal{B}}_2$ .

□

- Finally we get the chain of inequalities:

$$\begin{aligned}\left\|\mathbb{E}\left(f \mid \mathcal{B}_2'\right)\right\|_2^2 - \left\|\mathbb{E}\left(f \mid \mathcal{B}_2\right)\right\|_2^2 &= \left\|\mathbb{E}\left(f \mid \mathcal{B}_2'\right) - \mathbb{E}\left(f \mid \mathcal{B}_2\right)\right\|_2^2 \\ &= \left\|\mathbb{E}\left(g \mid \mathcal{B}_2'\right)\right\|_2^2 \\ &\geqslant \left\|\mathbb{E}\left(g \mid \widetilde{\mathcal{B}}_2\right)\right\|_2^2 \\ &\geqslant \left\|\mathbb{E}\left(g \mid \widetilde{\mathcal{B}}_2\right)\right\|_1^2 \\ &\geqslant c(\delta)\end{aligned}$$

by Pythagoras theorem, the definition of $g$ and the fact that $\mathcal{B}_2'$ refines $\widetilde{\mathcal{B}}_2$, and finally Cauchy Schwarz.

# Quadratic Koopman-von Neumann decomposition

**Theorem**

*Let $\left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$ be a quadratic factor with complexity at most $\left(d_1^{(0)}, d_2^{(0)}\right)$. Let $f : \mathbb{F}_5^n \to [-1, 1]$ be a function and let $\delta > 0$ be a parameter. "Then there is a quadratic factor $(\mathcal{B}_1, \mathcal{B}_2)$ of complexity at most $\left(d_1^{(0)} + O_\delta(1), d_2^{(0)} + O_\delta(1)\right)$ which refines $\left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$, and such that*

$$f = f_1 + f_2$$

*where*

$$f_1 := \mathbb{E}\left(f \mid \mathcal{B}_2\right)$$

*and*

$$\|f_2\|_{U^3} \leqslant \delta.$$

## Proof.

- Start with $(\mathcal{B}_1, \mathcal{B}_2) = \left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$. If
$$\|f - \mathbb{E}(f \mid \mathcal{B}_2)\|_{U^3} \geq \delta,$$
energy increment is applicable to the factor $(\mathcal{B}_1, \mathcal{B}_2)$.
- We will get $(\mathcal{B}_1, \mathcal{B}_2)$ whose complexity is increased by $(1, 1)$, and $\|\mathbb{E}(f \mid \mathcal{B}_2)\|$ is increased by $c(\delta)$.
- By the choice of non-decreasing $c$, this algorithm must be applicable at most $1/c(\delta)$ times.
- Algorithm is not applicable $\iff$ the conditions of energy increment are not satisfied.
- For some $\mathcal{B}_2$,
$$\|f - \mathbb{E}(f \mid \mathcal{B}_2)\|_{U^3} < \delta.$$

□

## Theorem (Arithmetic regularity lemma for $U^3$ -I )

*Let $\delta > 0$ be a parameter, and let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$be an arbitrary growth function (which may depend on $\delta$ ). Suppose that $n > n_0(\omega, \delta)$ is sufficiently large, and let $f : \mathbb{F}_5^n \to [-1, 1]$ be a function. Let $\left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$ be a quadratic factor of complexity $\left(d_1^{(0)}, d_2^{(0)}\right)$. Then there is $C = C\left(\delta, \omega, d_1^{(0)}, d_2^{(0)}\right)$ and a quadratic factor $(\mathcal{B}_1, \mathcal{B}_2)$ which refines $\left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$ and has complexity at most $(d, d)$, $d \leqslant C$, together with a decomposition*

$$f = f_1 + f_2 + f_3$$

*where*

$$f_1 := \mathbb{E}\left(f \mid \mathcal{B}_2\right), \\ \left\|f_2\right\|_2 \leqslant \delta$$

*and*

$$\left\|f_3\right\|_{U^3} \leqslant 1/\omega(d).$$

## Theorem (Arithmetic regularity lemma for $U^3$ -II )

*Let $\delta > 0$ be a parameter, and let $\omega_1, \omega_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be arbitrary growth functions (which may depend on $\delta$ ). Let $n > n_0(\delta, \omega_1, \omega_2)$ be sufficiently large, and let $f : \mathbb{F}_5^n \to [-1, 1]$ be a function. Let $\left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$ be a quadratic factor of complexity $\left(d_1^{(0)}, d_2^{(0)}\right)$. Then there is a quadratic factor $(\mathcal{B}_1, \mathcal{B}_2)$ with the following properties: (1) $(\mathcal{B}_1, \mathcal{B}_2)$ refines $\left(\mathcal{B}_1^{(0)}, \mathcal{B}_2^{(0)}\right)$; (2) The complexity of $(\mathcal{B}_1, \mathcal{B}_2)$ is at most $(d_1, d_2)$, where*

$$d_1, d_2 \leqslant C\left(\delta, \omega_1, \omega_2, d_1^{(0)}, d_2^{(0)}\right)$$

*for some fixed function $C$; (3) The rank of $(\mathcal{B}_1, \mathcal{B}_2)$ is at least $\omega_1(d_1 + d_2)$; (4) There is a decomposition $f = f_1 + f_2 + f_3$, where*

$$f_1 := \mathbb{E}(f \mid \mathcal{B}_2),$$
$$\|f_2\|_2 \leqslant \delta$$

*and*

$$\|f_3\|_{U^3} \leqslant 1/\omega_2(d_1 + d_2).$$

- The first proof is a result of the iterative application of Koopman-von Neumann decomposition.
- We get a small $\delta$ in terms of the complexity of the quadratic factor.
- The second proof utilizes the fact that every quadratic factor can be refined to a "high-rank" quadratic factor whose complexities are "close".

# Main Theorem

Theorem (Main Theorem)

*Let* $\alpha, \epsilon > 0 \in \mathbb{R}$. *Then* $\exists n_0 = n_0(\alpha, \epsilon)$ *with the following property: Suppose that* $n > n_0$ *and* $A \subseteq \mathbb{F}_5^n$ *has density* $\alpha$. *Then* $\exists d \neq 0$ *s.t.* $A$ *contains* $\geq (\alpha^4 - \epsilon)N$ *4APs with common difference d* ($N = |\mathbb{F}_5^n| = 5^n$).

- Applying Arithmetic regularity II to $1_A$ where $A \subset \mathbb{F}_5^n$. We get a decomposition $1_A = f_1 + f_2 + f_3$ where

$$f_1 := \mathbb{E}\left(f \mid \mathcal{B}_2\right),$$
$$\|f_2\|_2 \leqslant \delta$$

  and

$$\|f_3\|_{U^3} \leqslant 1/\omega\left(d_1 + d_2\right)$$

  and the quadratic factor $(\mathcal{B}_1, \mathcal{B}_2)$ is of complexity $(d_1, d_2)$, where $d_i < d_0(\alpha, \epsilon)$ and rank $r$ satisfying

$$r > 100(log(1/\epsilon) + \log(1/\alpha) + d_1 + d_2).$$

- The parameters $\delta$ and $\omega$ will be specified later.

- Let $r_1^T x, \ldots, r_{d_1}^T x$ be the linear functions in $\mathcal{B}_1$.
- Define $H := \langle r_1, \ldots, r_{d_1} \rangle^T$. Let $1_H$ be the characteristic function of $H$, and let $\mu_H$ be the normalised measure on $H$,

$$\mu_H := 1_H / \mathbb{E} 1_H$$

.

- What we will prove is:

$$\mathrm{E}_{x,d} 1_A(x) 1_A(x+d) 1_A(x+2d) 1_A(x+3d) \mu_H(d) \geqslant \alpha^4 - \epsilon,$$

which implies the main theorem for some $d \in H$ by an averaging argument.

- We split the left-hand side of (4.9) into 81 parts by substituting $1_A = f_1 + f_2 + f_3$.

- The terms containing $f_2$: Some of the terms are of the form

$$\mathbb{E}_{x,d}g_1(x)g_2(x+d)g_3(x+2d)g_4(x+3d)\mu_H(d),$$

where $g_1 = f_2$.

- Set $F(x) := \mathbb{E}_d g_2(x+\text{ d})\ g_3(x+2d)g_4(x+3d)\mu_H(d)$. It follows that

$$\begin{aligned}|\mathbb{E}_{x,d}g_1(x)g_2(x+d)g_3(x+2d)g_4(x+3d)\mu'_H(d)| &\le |\mathbb{E}_x g_1(x)F(x)| \\ &\le \|f_2\|_1 \le \|f_2\|_2\,,\end{aligned}$$

as $\|F\|_\infty \leqslant 1$. This proves the claim provided that $\delta \le \epsilon/200$.

- The terms containing $f_3$ can be bounded by $\epsilon/200$ using generalized von Neumann.

- Making the other 80 factors small, we focus on the main term:

$$\mathbb{E}_{x,d} f_1(x) f_1(x+d) f_1(x+2d) f_1(x+3d) \mu_H(d).$$

- For $f_1 : \mathbb{F}_5^n \to \mathbb{C}$ is a $\mathcal{B}_2$-measurable function then we write $\mathbf{f}_1 : \mathbb{F}_5^{d_1} \times \mathbb{F}_5^{d_2} \to \mathbb{C}$ for the function which satisfies

$$f_1(x) = \mathbf{f}_1(\Gamma(x), \Phi(x))$$

where $\Gamma(x) := (r_1^T x, ..., r_{d_1}^T x)$ and $\Phi(x) := (x^T M_1 x, ..., x^T M_{d_2} x)$.

- We will show

$$\mathbb{E}_{(a,b) \in \mathbb{F}_5^{d_1} \times \mathbb{F}_5^{d_2}} \mathbf{f}_1(a, b) = \alpha \left(1 + O\left(5^{2d_1 + 2d_2 - r/2}\right)\right).$$

- Clearly if the atoms were of the same size we had the equality without the error term.
- We will now study the atoms' size.

## Lemma

*Suppose that* $(\mathcal{B}_1, \mathcal{B}_2)$ *has mank at least* $r$*. Let* $(a, b) \in \mathbb{F}_5^{d_1} \times \mathbb{F}_5^{d_2}$*. Then the probability that a randomly chosen* $x \in \mathbb{F}_5^n$ *has* $\Gamma(x) = a$ *and* $\Phi(x) = b$ *is* $5^{-d_1-d_2} + O\left(5^{-r/2}\right)$*.*

## Proof.

$\Gamma(x) = a \iff r_j^T x = a_j$ for all $j$.

$$5^{-d_1-d_2}\mathbb{E}_x \prod_{i=1}^{d_1} \underbrace{\left(\sum_{\mu_i \in \mathbb{F}_5} \omega^{\mu_i\left(r_1^T x - a_j\right)}\right)}_{\begin{cases} 5 & \text{if } r_1^T x - a_j = 0 \\ 0 & \text{otherwise} \end{cases}} \prod_{j=1}^{d_2} \left(\sum_{\lambda_j \in \mathbb{F}_5} \omega^{\lambda_j\left(x^T M_j x - b_j\right)}\right),$$

□

- $$5^{-d_1-d_2}\sum_{\mu_i,\lambda}\omega^{-\lambda_1 b_1-\cdots-\lambda_{d_2}b_{d_2}-\mu_1 p_1-\cdots-\mu_{d_1}a_{d_1}}$$
$$\mathbb{E}_x\omega^{x^T\left(\lambda_1 M_1+\cdots+\lambda_{d_2}M_{d_2}\right)x+\left(\mu_1 r_1+\cdots+\mu_{d_1}r_{d_1}\right)^T x}.$$

- We have
$$\operatorname{rk}\left(\lambda_1 M_1+\cdots+\lambda_{d_2}M_{d_2}\right)\geq r.$$
By Gauss sum estimate every term in (4.1) in which the $\lambda_i$ are not all zero is bounded by $5^{-d_1-d_2-r/2}$.
- Among the remaining terms, the linear independence of the $r_i$ guarantees that the only term that does not vanish is that with $\mu_1=\cdots=\mu_{d_1}=0$.

## Lemma

*Suppose that* $(\mathcal{B}_1, \mathcal{B}_2)$ *has rank at least* $r$. *Suppose that* $\left(a^{(1)}, b^{(1)}\right), \ldots, \left(a^{(4)}, b^{(4)}\right) \in \mathbb{F}_5^{d_1} \times \mathbb{F}_5^{d_2}$. *Suppose that a 4 -term progression* $(x, x+d, x+2d, x+3d) \in \left(\mathbb{F}_5^n\right)^4$ *is chosen at random. If*

$$a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)} \text{ are in arithmetic progression}$$

*and*

$$b^{(1)} - 3b^{(2)} + 3b^{(3)} - b^{(4)} = 0$$

*then the probability that* $\Gamma(x+id) = a^{(i)}, \Phi(x+id) = b^{(i)}$ *for* $i = 1, 2, 3, 4$ *is* $5^{-2d_1-3d_2} + O\left(5^{-r/2}\right)$. *Otherwise, it is zero.*

Proof is omitted.

- We will continue to the proof of the main theorem. We have

$$\mathbb{E}_{(a,b)\in\mathbb{F}_5^{d_1}\times\mathbb{F}_5^{d_2}}\mathbf{f}_1(a,b)=\alpha\left(1+O\left(5^{2d_1+2d_2-r/2}\right)\right)$$

since $(\sum w_{a,b}\mathbf{f}_1(a,b))=\alpha$ where $w_{a,b}$ represents (number of elements in the atom specified by $a,b$)/(all elements). As

$$\max|(\sum w_{a,b}-\frac{1}{5^{d_1+d_2}})|\sum \mathbf{f}_1(a,b)\leq O(5^{2d_1+2d_2-r/2}).$$

($\mathbf{f}_1(a,b)\leq 5^{d_1+d_2}$ ), the result follows.

- Lastly we show

$$
\begin{aligned}
&\mathbb{E}_{x,d} f_1(x) f_1(x+d) f_1(x+2d) f_1(x+3d) \mu_H(d) \\
&= \mathbb{E}_{\substack{a \in \mathbb{F}_5^{d_1}, b^{(1)}, \ldots, b^{(4)} \in \mathbb{F}_5^{d_2} \\ b^{(1)} - 3b^{(2)} + 3b^{(3)} - b^{(4)} = 0}} \mathbf{f}_1\left(a, b^{(1)}\right) \mathbf{f}_1\left(a, b^{(2)}\right) \mathbf{f}_1\left(a, b^{(3)}\right) \mathbf{f}_1\left(a, b^{(4)}\right) \\
&+ O\left(5^{2d_1 + 3d_2 - r/2}\right).
\end{aligned}
$$