Density increment 2 - Roth

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The presentation here is a blend of Zhao and Kowalski, with a somewhat different style.

We wish to prove the following

Theorem 1 (Roth)**.** *Let $A \subseteq [1, N]$ be 3AP-free. Then $|A| \lesssim N/\log\log N$.*

Note that this is much weaker than what we got in Meshulam, which was of the form $N/\log N$, where N is the size of the underlying set ($\mathbb{F}_3^n$ in Meshulam). This is because the density increment we obtain will be only on a set of size $\sim N^s$.

Before starting the proof, observe that it suffices to prove the same result for $\mathbb{Z}/N\mathbb{Z}$ for odd N, because if $A \subseteq [1, N]$, then the canonical embedding of A into $\mathbb{Z}/(4N+1)\mathbb{Z}$ is also 3AP-free, so we only lose a constant of 5 or so. Let $\alpha = A/N$. As before, define $\Lambda(f, g, h) = \mathbb{E}_{x,d} f(x)g(x+d)h(x+2d)$, $\Lambda_3(f) = \Lambda(f, f, f)$. By Fourier analysis, $\Lambda(f, g, h) = \sum_r \hat{f}(r)\hat{g}(-2r)\hat{h}(r)$.

The proof follows more or less the same structure as Meshulam, but there are some technical details to take care of (non-existence of hyperplanes). In this instance, the density increment will be the following trichotomy:

- Either A contains a 3AP,
- $N \lesssim f(\alpha)$ for some decreasing f, or
- A has density $\geq \alpha + c(\alpha)$ for some increasing c on some AP P of odd size $\geq N^s$.

As one can see, AP's play the role of hyperplanes in this setting. However because of the lack of vector space structure, we lose an exponent when passing to an AP. Now we carry out the iteration argument. Different letters c may refer to different constants, but this should make no confusion.

Let $N_0 = N$, $A_0 = A$, $\alpha_0 = \alpha$. At each step i, we will either obtain an upper bound for α, or obtain $N_{i+1} \geq N_i^s$, $A_{i+1} \subseteq A \cap P$, $\alpha_{i+1} \geq \alpha + c(\alpha)$. Since 3APs are translation and dilation invariant, we can shift and dilate P to get $[1, N_{i+1}]$, and apply the trichotomy again. We now bound the number of steps we can apply. We assume $c(\alpha) \gtrsim \alpha^{1+t}$ for some positive t, which will be satisfied in our proof (we will have $c(\alpha) \gtrsim \alpha^2$), and $f(\alpha) = \alpha^{-c}$. By the argument in the Meshulam notes, the number of iterations $m \lesssim \alpha^{-t}/(1 - 2^{-t})$.

Thus the second option must eventually hold. When this happens, we will have $N^{s^m} \leq N_m \lesssim \alpha^{-c}$, so $s^m \log N \lesssim \log(1/\alpha)$, so $\log N \lesssim s^{-c\alpha^{-t}K(t)} \log(1/\alpha)$, so $\log\log N \lesssim_{s,t} \alpha^{-t}$. Thus $\alpha \lesssim (\log\log N)^{-1/t}$.

We have now seen how to use the trichotomy to get a bound for α. Now let us actually prove the trichotomy. The first step is, as in Meshulam, proving that a 3AP-free set has a large Fourier coefficient if the underlying group is sufficiently large. Throughout, we assume that N is odd.

Lemma 1. *If $A \subseteq \mathbb{Z}/N\mathbb{Z}$ is 3AP-free, then either $N < 5\alpha^{-2}$, or there is some $r \neq 0$ such that $|\hat{A}(r)| \geq \alpha^2/5$.*

Proof. Assume $N \geq 5\alpha^{-2}$, so that we have $\alpha^2 - 1/N \geq 4\alpha^2/5$. Since A is 3AP-free, $\Lambda_3(A) = \alpha/N$, since only the trivial progressions contribute. We have

$$\begin{aligned}
\alpha^3 - \alpha/N &= |\Lambda_3(\alpha) - \Lambda_3(A)| \\
&\leq |\Lambda(A-\alpha, A, A)| + |\Lambda(\alpha, A-\alpha, A)| + |\Lambda(\alpha,\alpha,A-\alpha)| \\
&= |\sum_r \widehat{A-\alpha}(r)\hat{A}(-2r)\hat{A}(r)| + |\sum_r \hat{\alpha}(r)\widehat{A-\alpha}(-2r)\hat{A}(r)| \\
&\quad + |\sum_r \hat{\alpha}(r)\hat{\alpha}(-2r)\widehat{A-\alpha}(r)| \\
&\leq \|\widehat{A-\alpha}\|_\infty (|\sum_r \hat{A}(-2r)\hat{A}(r)| + |\sum_r \hat{\alpha}(-2r)\hat{A}(r)| + |\sum_r \hat{\alpha}(-2r)\hat{\alpha}(r)|) \\
&\leq \|\widehat{A-\alpha}\|_\infty (\|\hat{A}\|_{\ell^2}^2 + \|\hat{A}\|_{\ell^2}\|\hat{\alpha}\|_{\ell^2} + \|\hat{\alpha}\|_{\ell^2}^2) \\
&= \|\widehat{A-\alpha}\|_\infty (\|A\|_2^2 + \|A\|_2\|\alpha\|_2 + \|\alpha\|_2^2) \\
&= \|\widehat{A-\alpha}\|_\infty (\alpha + \alpha^2 + \alpha^2) \leq 3\alpha\|\widehat{A-\alpha}\|_\infty, \quad (1)
\end{aligned}$$

by multilinearity, Cauchy-Schwarz and Parseval so so $\|\widehat{A-\alpha}\|_\infty \geq 4\alpha^2/15$, which completes the proof, since a nonzero FC of $A - \alpha$ is necessarily at a nontrivial character. □

Now we want to prove that this implies a density increment on a reasonably long AP. Assume that $\hat{A}(r) \geq \delta > 0$. If you think a little bit about the argument in Meshulam, it partitions the ambient space into very few parts (3) such that $\omega^{r\cdot}$ is constant on each part. In this setting this is hopeless because ω^{rx} can take much more values. However, we can try to make this hold only approximately on each part. If $P = i+q, i+2q, \ldots, i+kq$, then for any $x, y \in P$, we have $|e(rx/N) - e(ry/N)| = |1 - e(r(x-y)/N)| = |1 - e(rqt/N)|$ for some $|t| \leq k$. In order to ensure that this is small, we need $rqt \mod N$ to be small (or, equivalently, $\|rqt/N\|$ to be small, where norm denotes the distance to the closest integer) for all $|t| \leq k$. For any a, we have $rtq/N = at + qt(r/N - a/q)$, so $\|rqt/N\| = q|t|\|r/N - a/q\| \leq qk|r/N - a/q|$. Thus we need to find some a (and q, which we have not fixed yet, remember that the only thing fixed right now is r, which is the location of the large FC) such that a/q is close to r/N.

For any M, by Dirichlet we can find $q \leq M$ such that $|r/N - a/q| \leq 1/qM$. Thus $\|rqt/N\| \leq k/M$ for all $|t| \leq k$. Both k and M will depend on N, and we want $k/M \to 0$ as $N \to \infty$. Remember that k is the length of the AP, and M is an upper bound for the common difference. Keeping this in mind, suppose that we have fixed M and k depending on N, and obtained a, q by Dirichlet. We will now partition $\mathbb{Z}/N\mathbb{Z}$ into APs P_j with common difference q and sizes between k and $2k$ in the "obvious" way: Between each two consecutive multiples kqt and $kq(t+1)$ of kq except the last one less than N, take $kqt, kqt+q, \ldots, kqt+(k-1)q$, $kqt+1, kqt+1+q, \ldots, kqt+1+(k-1)q$, etc. The last part may not fill up k "slots" we need in our AP, so instead merge these parts into their corresponding APs between the previous multiples. In words perhaps this is not a very good explanation but if one thinks about it for a while one will see that this is actually a very obvious and natural way to do this.

Now, for each P_j and $x, y \in P_j$, we have $|e(rx/N) - e(ry/N)| \leq 20k/M$ (2 coming from $2k$, 10 coming from $|e(x) - e(y)| \leq 2\pi|x-y|$). We have finished the actual content of the argument, the remaining part being just technical considerations, which are nevertheless important. We prove the following

Lemma 2. *Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$ and $\hat{A}(r) \geq \delta > 0$ for some $r \neq 0$. Then either $N \lesssim \delta^{-6}$ or there is an AP P of odd length between $N^{1/3}$ and $2N^{1/3}$ such that $|A \cap P|/|P| \geq \alpha + \delta/5$.*

Proof. Assume $N \geq 160^6\delta^{-6}$. Then $40N^{-1/6} \leq \delta/4$. Choose $k = \lceil N^{1/3} \rceil + 1$, $M = N^{1/2}$ and let P_j be as in the discussion preceding the lemma. For each j, fix some $y_j \in P_j$. In view of the previous discussion, we have

$$\begin{aligned}
\delta \leq |\hat{A}(r)| &\leq \frac{1}{N}\sum_j \left|\sum_{x\in P_j}(A-\alpha)(r)e(rx/N)\right| \\
&= \frac{1}{N}\sum_j \left|\sum_{x\in P_j}(A-\alpha)(r)(e(ry_i/N) - e(ry_i/N) + e(rx/N))\right| \\
&\leq \frac{1}{N}\sum_j \left(\left|\sum_{x\in P_j}(A-\alpha)(r)\right| + \sum_{x\in P_j}|e(ry_i/N) - e(rx/N)|\right) \\
&\leq \frac{1}{N}\sum_j \left(\Big||A\cap P_j| - \alpha|P_j|\Big| + 20k|P_j|/M\right) \\
&= \frac{1}{N}\sum_j \Big||A\cap P_j| - \alpha|P_j|\Big| + 20k/M \\
&\leq \frac{1}{N}\sum_j \left(\Big||A\cap P_j| - \alpha|P_j|\Big| + |A\cap P_j| - \alpha|P_j|\right) + 40N^{-1/6}. \quad (2)
\end{aligned}$$

Thus $3\delta N/4 \leq \sum_j \left(\Big||A\cap P_j| - \alpha|P_j|\Big| + |A\cap P_j| - \alpha|P_j|\right)$, so by pigeonhole (notice that we are using the same trick as in Meshulam here, $N = \sum_j |P_j|$) there is some j with $|A\cap P_j| - \alpha|P_j| \geq (\alpha + 3\delta/8)|P_j|$, which completes the proof.

A very minor technical point is that $|P_j|$ needs to be odd (we need to ensure that multiplication by 2 is a bijection on $\mathbb{Z}/N\mathbb{Z}$), but this is easily achieved as if $|P_j|$ is odd, we can just delete an element of P_j not contained in A (if this is not possible then the relative density is 1 and we can delete an arbitrary element with no change to relative density) and increase the density. Now the size of P_j has decreased but is certainly still above $N^{1/3}$, so we are done. □

Now apply this lemma with $\delta = \alpha^2/5$. We obtain the trichotomy with $s = 1/3$, $f(\alpha) = C\alpha^{-12}$, $c(\alpha) = k\alpha^2$ for some absolute constants C, k, which completes the proof of Roth's theorem.