Density increment 1 - Meshulam

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The presentation here is mostly from Zhao's book with some modifications. We wish to prove the following

Theorem 1 (Meshulam)**.** *Let* $A \subseteq \mathbb{F}_3^n$ *be a set not containing any triple* $(a, a+d, a+d+d)$ *with* $d \neq 0$*. Then* $|A| \lesssim 3^n/n$*.*

The argument extends easily to any $\mathbb{F}_p^n$.

First we provide a rough sketch of the proof. There are three steps, which I like to call (for obvious reasons)

1. 3AP-free $\implies$ large FC
2. Large FC $\implies$ density increment on hyperplane
3. Iterate.

The hardest step is the first one probably, because one needs to avoid a mistake. Second step is just writing down in math language the fact that the Fourier transform of a function at a point measures the irregularity of the distribution of said function on the cosets of the orthogonal complement of the point. Third step is observing that the density cannot increase forever since it is bounded by 1. Enough talk, let us get to work.

We define the 3AP counting operator $\Lambda(f,g,h) = \mathbb{E}_{x,d} f(x)g(x+d)h(x+d+d)$, $\Lambda_3(f) = \Lambda(f,f,f)$. By Fourier analysis, we have $\Lambda(f,g,h) = \sum_r \hat{f}(r)\hat{g}(r)\hat{h}(r)$ (this is more or less the only difference between the cases $p=3$ and $p>3$).

We start with the set A, which we hypothesised to contain no 3AP. We write $\alpha = |A|/3^n$ the density of A.

In this problem, density increment roughly corresponds to the following trichotomy (we can and will make all these statements more quantitative):

- Either A contains a 3AP,
- $3^n \lesssim f(\alpha)$ for a decreasing f, or
- A has density $\geq \alpha + c(\alpha)$ when restricted to a hyperplane, where c is increasing.

Let us see why this trichotomy is useful. We start with our set A. Since A does not contain a 3AP, either the second or the third alternatives hold. If the third holds with the hyperplane V, then we consider $A' = A \cap V$. By hypothesis A' is also 3AP-free, and we can translate A' and V to make V a vector subspace (since 3APs are translation invariant, we can do this and A' will still be 3AP-free), and now we have the trichotomy again with $V \simeq \mathbb{F}_3^{n-1}$ and A' in place of G and A. We iteratively run this argument, starting with $\alpha_0 = \alpha, A_0 = A$. At each step we obtain A_i with density $\geq \alpha_{i-1} + c(\alpha_{i-1}) \geq \alpha + (i-1)c(\alpha)$. This can go on for at most $m \lesssim 1/c(\alpha)$ steps, since $a_i \leq 1$. At that point, by the trichotomy we must have $3^{n-m} \lesssim f(\alpha_m) \lesssim f(\alpha)$, so $n \lesssim 1/c(\alpha) + \log f(\alpha)$, which translates to an upper bound for α. Note that the iteration does not have to end only when the density reaches 1. It ends whenever we fail to get a density increment on a hyperplane, and we found that this has to happen some time in the first $\sim 1/c(\alpha)$ steps.

Actually there is a more clever way to do this iteration, but we need to assume a mild growth condition on c: $c(\alpha) \gtrsim \alpha^{1+s}$ for some $s > 0$ (weaker hypotheses might work with somewhat different bounds but this is the cleanest and this is satisfied in our case so I did not bother to make the condition weaker). To get from α to 2α, we need $\leq \alpha/c(\alpha)$ iterations, to get from 2α to 4α we need $\leq 2\alpha/c(2\alpha)$ iterations, et cetera. α can double at most $\lesssim \log(1/\alpha)$ times, so the number of iterations is bounded by

$$\sum_{j \leq 5\log(1/\alpha)} 2^j\alpha/c(2^j\alpha) \lesssim \sum_j 2^j\alpha/(2^j\alpha)^{1+s} \lesssim \alpha^{-s}K(s), \tag{1}$$

where $K(s)$ is $1/(1-2^{-s})$.

After this rather abstract discussion taking the trichotomy for granted, let us actually prove it. I think one will find it instructive to read the previous discussion again after this proof.

Now we prove step 1 (3AP-free $\implies$ large FC).

Lemma 1. *If $A \subseteq \mathbb{F}_3^n$ is 3AP-free, then either $3^n < 2\alpha^{-2}$, or $\max_{r\neq 0} |\hat{A}(r)| \geq \alpha^2/2$.*

Proof. Assume $3^n \geq 2\alpha^{-2}/2$, so that $\alpha^3 - \alpha/3^n \geq \alpha^3/2$. Since A is 3AP-free, $\Lambda_3(A) = \alpha/3^n$ (only the trivial progressions contribute). Then, by splitting the characters into zero and nonzero and using Parseval we get

$$\alpha^3/2 \leq \alpha^3 - \alpha/3^n = |\Lambda_3(A) - (\mathbb{E}A)^3| \leq \sum_{r\neq 0} |\hat{A}(r)|^3 \tag{2}$$

$$\leq \max_{r\neq 0} |\hat{A}(r)| \sum_{r\neq 0} |\hat{A}(r)|^2 \leq \max_{r\neq 0} |\hat{A}(r)| \|A\|_2^2 = \alpha \max_{r\neq 0} |\hat{A}(r)|. \tag{3}$$

Dividing by α, we obtain the result. □

Note that if instead of using Parseval we just bounded (2) trivially by the maximum, then we would obtain something like $\max_{r\neq 0} |\hat{A}(r)| \gtrsim \alpha^3/3^{n/3}$, which is way too small to be useful, as we do not have lower bound for $1/3^n$ in terms of α. This is the mistake I mentioned previously. As an exercise one may try to iterate this bad bound, and will see that it leads nowhere (the bounds I got were worse than trivial, but I did not attempt to optimize the argument).

Now we prove step 2 (large FC $\implies$ density increment on a hyperplane). Combining this with step 1, we obtain the trichotomy with $f(\alpha) = \alpha^{-2}$ and $c(\alpha) = \alpha^2/4$. Let us first see how we can obtain the theorem from this. First, for illustrative purposes, the naive iteration: $n \lesssim \alpha^{-2} + \log(\alpha^{-2}) \lesssim \alpha^{-2}$, so $\alpha \lesssim n^{-1/2}$. The clever iteration, similarly, $n \lesssim \alpha^{-1}$, so $\alpha \lesssim 1/n$, as claimed.

Lemma 2. *If $|\hat{A}(r)| \geq \delta > 0$ for some $r \neq 0$, then A has density $\geq \alpha + \delta/2$ on some hyperplane.*

I think one should meditate on this statement a little bit. In the contrapositive this reads as follows: if A has density $< \alpha + \delta/2$ on each hyperplane, then $|\hat{A}(r)| < \delta$ for all $r \neq 0$. This is a manifestation of the fact that the Fourier coefficient at r measures how irregularly distributed A is on cosets of $r^{\perp}$.

Proof. Let α_i be the density of A on $\{x : r \cdot x + i = 0\}$ (addition is mod 3, and $i = 0, 1, 2$). Clearly $\alpha = \sum_i \alpha_i/3$, and by an easy calculation $3\hat{A}(r) = \alpha_0 + \alpha_1\omega + \alpha_2\omega^2$. Then, by adding $0 = \alpha(1 + \omega + \omega^2)$,

$$3\delta \leq |\sum_i \alpha_i \omega^i| = |\sum_i (\alpha_i - \alpha)\omega^i| \leq \sum_i |\alpha_i - \alpha| \tag{4}$$

Now this only gives a hyperplane where the density of A is at least δ apart from α. There is a cute trick to guarantee that we get an increment. We add $0 = \sum_i \alpha_i - \alpha$, so that (4) $= \sum_i |\alpha_i - \alpha| + (\alpha_i - \alpha)$. Thus, by pigeonhole there is some i such that $|\alpha_i - \alpha| + \alpha_i - \alpha \geq \delta > 0$. Since LHS is positive, we must have $\alpha_i > \alpha$, and division gives the desired result. This also completes the proof of Meshulam's theorem. □