Amenable groups and Følner sequences

Bora Çalım

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- For $s \in G$, write $sf(\cdot)$ for the function $f(s^{-1}\cdot)$ on G. A mean σ is left-invariant if $\sigma(sf) = \sigma(f)$ for all $s \in G$ and all f.

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- This is actually anachronistic – amenability was first discovered through paradoxicality, more on that later

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- If $N \lhd G$ and N and G/N are amenable, then G is amenable (again combine the means in the obvious way) – corollary: direct product of two amenable groups is amenable
- If $H_1 \leq H_2 \leq H_3 \leq \ldots$, each H_i is amenable and $G = \bigcup H_i$, then G is amenable (Let σ_i be a left-inv. mean on H_i. Extend trivially on G. Then each σ_i has norm at most one, so by Banach-Alaoglu they have a weak*-convergent subsequence, and the weak* limit of this is a left-inv. mean on G). – corollary: if each finitely generated subgroup of G is amenable, then G is amenable

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- Very basic example: $[-n, n] \subseteq \mathbb{Z}$.
- In the ISEM it was proved that every abelian group has a Følner sequence.
- A question arises: does every group have a Følner sequence? In order to answer this, we need more definitions.

Paradoxicality

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- The free group on two symbols $F_2 = \langle a, b \rangle$ is paradoxical: Consider the sets W_a, $W_{a^{-1}}$, W_b, $W_{b^{-1}}$ of the elements starting with the symbol in the subscript. F_2 is the union of these four sets, and $F_2 = W_a \sqcup W_a^c \sim W_a \sqcup a^{-1} W_a^c = W_a \sqcup W_{a^{-1}}$ and by the same reasoning $F_2 \sim W_b \sqcup W_{b^{-1}}$, so F_2 is paradoxical.

Connecting the different notions

Theorem

Let G be a countable discrete group. Then the following are equivalent:

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- Corollaries: Any group containing F_2 is not amenable ($SO(3)$!). Solvable, nilpotent, abelian groups are all amenable.

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- There exist groups of exponential growth but are amenable (Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$)
- There exist non-amenable groups not containing F_2 (Olshanskii)

Proof of theorem - amenable $\implies$ non-paradoxical and Følner $\implies$ amenable

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- Let F_n be a Følner sequence on G, and write $f_n = \mathbf{1}_{F_n}/|F_n|$. All of these are in $\ell^1 \subseteq (\ell^\infty)^*$, and have norms bounded by 1, so by Banach-Alaoglu f_n has a subsequence which weak*-converges to some $f \in (\ell^\infty)^*$.

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- This f is a left inv. mean for G.

Proof of theorem - no Følner $\implies$ paradoxical - step 1 - a set that doubles all finite sets

- I claim that there is a finite set S that expands *every* finite set by a factor of 2, i.e., $|SF| > 2|F|$ for all finite F. Observe: it suffices to prove this with any $\lambda > 1$ in place of 2 (we can take S^k for large k).

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- Assume that this does not hold. Consider finite sets S_n increasing to G and containing the id (so that $S_nF \supseteq F$). Then for all n, I can find a finite F_n such that $|S_nF_n| < (1 + 1/2n)|F_n|$. Now, for a fixed $s \in G$, S_n contains s and s^{-1} for $n > N_s$. Now, for such n,

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- $|sF_n \Delta F_n| \leq |sF_n \setminus F_n| + |F_n \setminus sF_n| = |sF_n \setminus F_n| + |s^{-1}F_n \setminus F_n|$
$\leq 2|S_nF_n \setminus F_n| \leq |F_n|/n$, so F_n is Følner.

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- This is an example of the interplay between the growth of product sets and the Følner condition. (Remark: How does S look like?)

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- Consider $G_1 \sqcup G_2$, a disjoint union of two copies of G. Consider finite sets $A_{g,i}$ where $g \in G$ and i denotes which of G_1 and G_2 it comes from, that satisfy $|\bigcup_{x \in K_1} A_{x,1} \cup \bigcup_{x \in K_2} A_{x,2}| \geq |K_1| + |K_2|$ for all $K_i \subseteq G$; and $A_{g,i} \subseteq Sg$ for all g.

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- $A_{g,i} = Sg$ satisfies these conditions: $|\bigcup_{x \in K_1 \cup K_2} Sx| = |S(K_1 \cup K_2)| \geq 2|K_1 \cup K_2| \geq 2\max(|K_1|, |K_2|) \geq |K_1| + |K_2|$.

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- Then by Zorn we can find a minimal family $M_{g,i}$ satisfying these (since for every chain its indexwise intersection satisfies the conditions).

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- Recall: We found minimal $M_{g,i}$ such that $|\bigcup_{x\in K_1} M_{x,1} \cup \bigcup_{x\in K_2} M_{x,2}| \geq |K_1| + |K_2|$ for all finite $K_i \subseteq G$; and $M_{g,i} \subseteq Sg$ for all g

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- $|K_1'| + |K_2'| \geq |R_1| + |R_2| = |R_1 \cup R_2| + |R_1 \cap R_2| \geq |M_{g,i} \cup \bigcup_{(h,j)\in K_1'\cup K_2'} M_{h,j}| + |\bigcup_{(h,j)\in K_1'\cap K_2'} M_{h,j}| \geq 1 + |K_1' \cup K_2'| + |K_1' \cap K_2'| = 1 + |K_1'| + |K_2'|$, contradiction! (key role: R_i do not satisfy the condition but from operations on them we can obtain expressions involving $M_{h,j}$, allowing us to exploit).

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- But: $\bigsqcup_s sC_s \sqcup \bigsqcup_s sDs = G$, so we have a paradoxical decomposition!

Recalling some ergodic theory – and where we are going

- Recall (from the lectures plus the equivalence we proved): Let G act pmp on X. Then for $f \in L^2(X)$, $\frac{1}{|F_i|}\sum_{s\in F_i} sf$ converges in L^2 (in fact, it is the projection of f in the subspace of G-inv. functions).

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Theorem

Let G act pmp on X, and F_n be a Følner sequence such that $|\bigcup_{k<n} F_k^{-1} F_n| \leq C|F_n|$. Then, for any $f \in L^1(G)$, $\lim_{n\to\infty} |F_n|^{-1} \sum_{s \in F_n} f(s^{-1}x)$ exists a.e.

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- Følner sequences that satisfy this condition are called *tempered*.

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Theorem (Emerson)

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- We will take $f(z) = z^{-1/2}$.

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- Consider I_n as I_1 starting from 0, and has length $1/1$; I_n starts at the end of I_{n-1}, and has length $1/n$. Then every $z \in \mathbb{T}$ is contained in infinitely many I_n.

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- First term $\to 0$, second term is $\leq 2(3(n-1)! + n!/n^{1/4})/2n! \to 0$. So we have a Følner sequence.

Temperedness condition is necessary - proving the pointwise divergence

- The key point is that elements of D_n make the values of f very large ($f(m \cdot z) \geq (n/2)^{1/2}$ for $z \in I_n$, $m \in D_n$). Fix z. Then for any n with $z \in I_n$

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- This proves that the $\limsup$ is infinite.

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- $|\tilde{F}_j F_{n_j}| \leq |F_{n_j}| + |F_{n_j} \Delta \tilde{F}_j F_{n_j}| \leq |F_{n_j}| + \sum_{g \in \tilde{F}_j} |F_{n_j} \Delta g F_{n_j}| \leq 2|F_{n_j}|$.